TWISTED QUARTIC CURVES OF THE FIRST SPECIES AND CERTAIN COVARIANT QUARTICS*

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The advantages of representing points of a plane cubic curve by values of an elliptic integral of the first kind are generally understood. That method gives the simplest exposition of the configurations associated with the inflexional points; of the theory of residuation; of the Steinerian Polygons; and of the linear construction of the cubic from pairs of conjugate points, which we owe to Schroeter. The plane cubic is the first curve (properly so called) in the indefinitely extended series of Elliptic Normal Curves—so named by Klein. The next in order is the twisted quartic in three dimensions, the intersection-line of two quadric surfaces. The application of elliptic parameters to such quartics is perhaps less familiar, though even more advantageous than in the lower case. What, for example, could be more elegant than Harnack's linear construction of conjugate triplets of points upon the curve from two given triplets of the same kind? Or where can better geometric illustrations be found for the problems of multiplication and division of elliptic arguments than in the relations of systems of quadric surfaces in the sheaf passing through the curve?

I propose here to use this parametric representation of points on the curve as an aid to proving the existence of certain irrational covariant curves and rational covariant surfaces.

1. The Index of a Quadric Surface, and of a Derived Quartic. The points in which a plane touches the curve and cuts it again are denoted, according to Abel's Theorem, by the parameters

$$-a+b, \quad a, \quad -a-b,$$

since

$$(-a+b)+(2a)+(-a-b)=0.$$

A chain of points such that a tangent plane in each can cut the curve in

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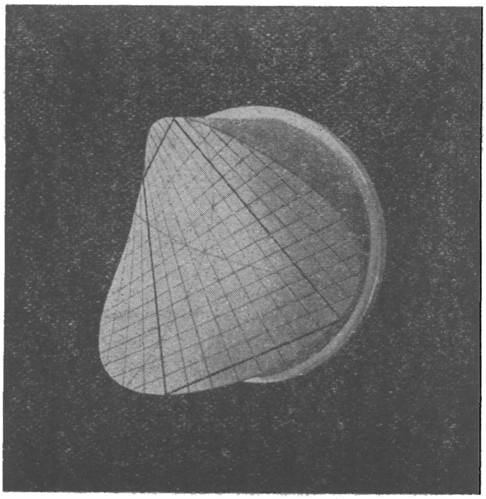
the point which precedes and the point which follows may therefore be denoted by the parameters :

$$\begin{array}{ccccccccccc} \ldots & & a-2b & & a & & a+2b & & a+4b & & \ldots \\ \ldots & -a+3b & & -a+b & & -a-b & & -a-3b & & \ldots & . \end{array}$$

Lines joining two consecutive points, with the positive coefficient of b the greater, will cut every line joining consecutive points of the opposite sort; for the sum of the four arguments will be zero. The two sorts of join-lines are therefore generators in the two reguli of a quadric surface containing the curve. Upon the quadric we shall consider a gauche polygon inscribed in the



curve. Depending on the nature of the quadric which supports it, this polygon will prove to be either indefinite or closed. Since generators of the same sheaf meet nowhere, a polygon can close only with some even number of sides, $2n$. If it closes with $2n$ sides, let us say that n is the *index* of the supporting quadric with respect to the curve. Among the sheaf of supporting quadrics, those of finite index will be rare, those of no index or of infinite index, the rule.

It is of course well known that the index is the same, whatever point of the curve be chosen as the first point of the polygon, and whichever of the

two generators meeting there be taken as the first side; for the condition of closing is

$$a + 2nb \equiv a \qquad \text{(modulis the periods } \omega_1,\ \omega_2\text{)},$$

or

$$2nb = m_1\omega_1 + m_2\omega_2 = P,$$

whence

$$b = \frac{P}{2n};$$

that is, the index is n if the argument b is a primitive $2n$th part of a period.

The Vossian quadrics are the six whose index is 2; which support therefore inscribed quadrilaterals. A model, at least approximately correct, of one such surface is Brill's No. 15, 3rd Series, showing a hyperbolic paraboloid on which the quartic curve is cut out by a cylinder. (See figure.) For each kind of quartic curve of the first species the number of real Vossian quadrics is discussed by Harnack, in volume 12 of the *Mathematische Annalen.*

A new system of curves is discovered if, while regarding a particular quadric of the sheaf, whose index we may think of as infinite, we suppose the sides of every inscribed gauche polygon to be produced indefinitely. Starting from any point, the sides may be called the first odd side, first even side, second odd, second even, etc. The intersection of a first odd side with its second even side has for its locus another curve upon the same surface;—an algebraic curve, for the determining conditions are algebraic, and of the fourth order, since each generator cuts it twice. Call this the C_2 of the given surface, and by analogy call the fundamental quartic the C_1. Now in the same way define C_3 as the locus of the intersection of each first odd side with its third even side, and so may be defined an endless series of quartic curves: C_4, C_5, . . ., upon the surface of infinite index.

Suppose however that the quadric has finite index, n. Then evidently the C_2 and the C_{n-2} will be the same quartic, also C_3 and C_{n-3}, etc. Hence the index of any curve of the system may be taken less than $(n + 1)/2$, where n is the index of the supporting quadric. There is a distinction between quadrics of even index and those of odd index; for the latter contain among the concomitant quartics always one of index $(n + 1)/2$ which degenerates evidently into a doubly-counting plane section, having however as many tangent generators as the C_1. As regards the number of derivative quartic curves on any surface, and the derivatives of derivatives, the analogy of regular polygons inscribed in a circle and the related star polygons is sufficiently close to allow trustworthy use of geometric imagination.

One thing it concerns us to note: that the definitions of auxiliary figures involve only projectively invariant relations—tangency, coplanarity, intersections; and that beyond those the relations of curve to surface, and of curve to curve in the surface, are denoted by pure numbers; *that therefore all relations here discussed are invariant with respect to collineation in three-dimensional space.*

2. Variable C_2, C_3, C_n as Characteristics of new Covariant Surfaces. On every quadric containing the fundamental quartic curve there exist quartics corresponding to all integral indices. On some the series is finite, and the numbers recur in regular order as their index describes the series of natural numbers; the six Vossian Quadrics showing only the fundamental quartic itself as corresponding to every index. We may safely assume that continuous variation of the supporting quadric causes each derivative quartic C_2, C_3, . . . C_n to describe a continuous locus, an algebraic surface. It is possible to calculate the degree and predict the singular points of each such surface in advance of the labor of finding its equation. One example will show the method; we shall discuss the locus of the first derivative quartic C_2.

Two surfaces of the sheaf have in common no points save those of the fundamental quartic. Hence the locus of the C_2 can have no conical points or other singular points outside the C_1. Each secant of the C_1 intersects its own C_2 in two points, and meets no other C_2. Hence the order of the locus is 2, increased by the number of intersections to be counted upon the C_1. Now the only surfaces on which the C_1 is at the same time a C_2 are the Vossian surfaces, six in number. *Hence the fundamental quartic must be a sextuple line upon the desired locus, and in it the locus must be tangent to each of the Vossian surfaces.* The order is then: $2 + 6 \cdot 2 = 14$. There are 12 quadrics having index 3 with respect to the curve. On each of these the C_2 is a plane conic twice counted. Twelve quadric surfaces of the sheaf therefore are tangent to the 14-ic locus along as many conics.

The next surface, the locus of curves C_3, is of order 38, and has the C_1 for an 18-fold line, along which it is tangent to the 6 Vossians and the 12 quadrics of period 3 respectively. The numerical characteristics increase rapidly. To extend the list could offer no difficulty. From the considerations 1) that each locus of a C_n is uniquely determinate and 2) that it is covariantly connected with the fundamental curve and not dependent upon any one quadric of the sheaf in distinction from any other, we draw the conclusion:

There exist simultaneous covariants which are combinants of the sheaf of quadrics containing the fundamental quartic, possessing the orders and singularities computed by the foregoing method.

3. On the Calculation of Equations of Derivative Quartics. The fundamental C_1 is projected from four points by cones. From one of these points the generators of any one of the quadrics are projected as tangents to a second cone. The inscribed polygon is projected into any plane as a polygon inscribed in one conic and circumscribed about a second. Salmon has given formulae for the locus of intersection of any first side with its third, fourth, fifth, etc.; and they appear as conics of a sheaf, whose equations are rational covariants of the equations of the two conics. To reckon the equations for a C_2, it would be best to assume the C_1 given by two quadric equations in canonical form:

$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0,$$
$$\Phi = A_1x_1^2 + A_2x_2^2 + A_3x_3^2 + A_4x_4^2 = 0.$$

For any quadric of the sheaf: $F + \lambda\Phi = 0$, the projection of its contour from one of the four cone-vertices is readily written out, together with the equation of the conic into which the quartic is projected. Then by Salmon's rules the trace of the C_2 can be found, and between its equation and $F + \lambda\Phi = 0$ the parameter λ may be eliminated. The eliminant will contain an unsymmetrical factor, which may be thrown out by the usual process, leaving the equation of the surface whose characteristic is a C_2. Similarly any other C_n and its locus could be represented algebraically, and, if desired, in terms of fundamental combinants in case of the surfaces.

The most inviting problem connected with these associated quartic curves is perhaps this: upon a quadric of given index, to find what relation subsists either between the elliptic moduli of the C_1 and the C_n, or between the invariant anharmonic ratios belonging to these two quartic curves.

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